International Journal of Theoretical Physics, Vol. 39, No. 1, 2000

Quantum Cosmology with Hyperbolic Potential

Subenoy Chakraborty[1] and Nabajit Chakravarty[2]

Received May 6, 1999

For an FRW model with a minimally coupled scalar field having hyperbolic (exponential) potential we evaluate the wave function both by solving the Wheeler–Dewitt (WD) equation and by evaluating the path integral. The WD equation is solved in configuration as well as in momentum space, while the path integral is evaluated by dividing the lapse integral into a number of pieces.

1. INTRODUCTION

The FRW model with a minimally coupled scalar field having hyperbolic (exponential) potential has been studied in recent years by Marugan and coworkers.[1,2] Gray *et al.*[1] studied this model for quantum cosmology with geometrodynamic variables and evaluated the wave function by a path integral formalism. Moreover, Marugan[2] developed a nonperturbative quantization program formulated by Ashtekar and Tate.[3] He also constructed reality conditions in quantum cosmology which select the quantum theory associated with a specific section of the complex phase space and also determine the proper inner product satisfying the adjointness condition.

In this paper, we perform a transformation of variables (including the field variable) which transforms the geometrodynamic variables to Ashtekar variables so that the constraint equation becomes simple in form. Then we evaluate the wave function both by solving the WD equation and by the path integral formalism.

[1] Department of Mathematics, Jadavpur University, Calcutta-32, India.
[2] Meteorological Observatory, Diamond Harbour, Indian Meteorological Department, India, 743331.

0020-7748/00/0100-0153$18.00/0

2. THE WAVE FUNCTION IN THE PATH INTEGRAL FORMALISM

The metric ansatz for a family of homogeneous and isotropic space-time models (with proper scaling of the lapse function) can be written as

$$dS^2 = -N^2 \frac{dt^2}{e^{2\alpha}} + e^{2\alpha}\, d\mathrm{O}_3^2 \tag{1}$$

Here e^{α} is the scale factor of the model, $N(t)$ is the rescaled lapse function, and $d\mathrm{O}_3^2$ is the metric of the unit three-sphere. For the minimally coupled scalar field $\emptyset(t)$, we consider a potential of the form

$$V(\emptyset) = a \cosh 2\emptyset + b \sinh 2\emptyset \tag{2}$$

with a, b real constants ($a = -b$ gives an exponential potential).

Now, using the transformation of variables $(\alpha, \emptyset) \rightarrow (x, y)$ defined by[2]

$$u = e^{2\alpha} \cosh 2\emptyset, \qquad v = e^{2\alpha} \sinh 2\emptyset \tag{3}$$

we find that the only constraint of the system (namely the scalar constraint) takes the following simple form in phase space:

$$H \equiv \frac{1}{2}\left[-4(\pi_u^2 - \pi_v^2) + au + bv - 1\right] = 0 \tag{4}$$

Here π_u and π_v are the momenta canonically conjugate to u and v.

In the path integral formalism, the wave function of the universe can be estimated by

$$\begin{aligned} \psi &= \int dN \int D\pi_u\, D\pi_v\, Du\, Dv\, \exp[-I(u, \pi_u, v, \pi_v)] \\ &\equiv \int dN \cdot \psi(N) \end{aligned} \tag{5}$$

Here the choice of contour for $\psi(N)$ can be indicated by the WD equation. But the lapse integral will not always be convergent. So we first divide the lapse interval into a finite number $(M + 1)$ of pieces and then integrate over (u, π_u) and (v, π_v). Hence we have[4,5]

$$\begin{aligned} \psi(N) = \lim_{M\to\infty} \int \prod_{i=1}^{M} du_i\, dv_i \cdot \prod_{i=0}^{M} \frac{d\pi_{u_i}}{2\pi}\frac{d\pi_{v_i}}{2\pi} \\ \times \exp\left\{ -\Delta N \sum_{i=0}^{L} \right. \\ \times \left. \left[\pi_{u_i}\frac{u_{i+1} - u_i}{\Delta N} + \pi_{v_i}\frac{v_{i+1} - v_i}{\Delta N} + 2\pi_{u_i}^2 - 2\pi_{v_i}^2 - \frac{a}{2}u_i - \frac{b}{2}v_i + \frac{1}{2} \right] \right\} \end{aligned} \tag{6}$$

with $\Delta N = N/(M + 1)$, $N_i = iN$ $(i = 1, 2, \ldots)$, $u_i = u(N_i)$, $v_i = v(N_i)$, $\pi_{u_i} = \pi_u(N_i)$, and $\pi_{v_i} = \pi_v(N_i)$. We note that the integrals over π_{u_i} and π_{v_i} are Gaussian in form. So after performing these Gaussian integrals the integrals over u_i and v_i are the standard integrals and we get

$$\psi = \int \frac{dN}{N^{3/2}} \exp(-I_0) \tag{7}$$

where

$$I_0 = -\frac{1}{8N} \{(u'' - u')^2 - (v'' - v')^2\} - N \left\{ \frac{a}{4} (u' + u'') + \frac{b}{4} (v' + v'') + 1 \right\}$$

Here (u', u'') and (v', v'') are the values of u and v at the two ends of the lapse variable.

Now, if we take the range for N-integration to be $(-\infty, \infty)$ (for which ψ is a solution to the WD equation), then the integral in (7) diverges. (If we consider a semiinfinite range for the lapse function, then ψ is a Green's function for the WD operator). So using the new regularization technique[(4)] we introduce a complex parameter μ as the scale factor for the lapse function to make the lapse integral convergent and perform the lapse integral. Then we take the analytic continuation $\mu \to 1$. The explicit expression for the wave function is

$$\psi = \sqrt{\frac{\pi}{B}} \exp(-2\sqrt{AB}) \tag{8}$$

where

$$A = -\left\{ \frac{a}{4} (u' + u'') + \frac{b}{4} (v' + v'') + 1 \right\} \tag{9}$$

and

$$B = \frac{1}{8} \{(v'' - v')^2 - (u'' - u')^2\}$$

Note that if instead of choosing the straight contour we consider a branch cut, we also get a solution that is zero.

3. THE WHEELER–DEWITT EQUATION AND THE SOLUTION

The WD equation is obtained from the Hamiltonion constraint by converting the canonical variables into the operator form. So from (4) the WD equation in momentum representation takes the form

$$\left[ia \frac{\partial}{\partial x} + ib \frac{\partial}{\partial b} + 4(y^2 - x^2) - 1 \right] \text{Ø}(x, y) = 0 \tag{10}$$

with $x = \pi_u$, $y = \pi_v$. One can easily check that it has a solution,

$$\text{Ø}(x, y) = \exp\left[\frac{4i}{3} \left(\frac{y^3}{b} - \frac{x^3}{a} \right) - \frac{ix}{a} \right\} \tag{11}$$

(with a constant multiplicative factor).

On the other hand, the Wheeler–DeWitt equation in configuration space can be written as

$$\left[-4 \frac{\partial^2}{\partial u^2} + 4 \frac{\partial^2}{\partial u^2} + au + bv - 1 \right] + (u, v) = 0 \tag{12}$$

Now, using separation of variables

$$\psi(u, v) = U(u) \cdot V(v) \tag{13}$$

we have the differential equations for U and V

$$4 \frac{d^2 U}{du^2} - a \left\{ u + \frac{k - 1/2}{a} \right\} U = 0 \tag{14}$$

and

$$4 \frac{d^2 V}{dv^2} + b \left\{ v - \frac{k + 1/2}{b} \right\} V = 0 \tag{15}$$

with K as the separation constant.

Equations (14), (15) are in identical form to the differential equation that we consider for the WKB approximation in quantum mechanics. So the solution for this type of equation has been well studied in quantum mechanics and its asymptotic behavior discussed.

For future work, it will be interesting to relate all these wave functions through integral transforms and the proper choice of the boundary conditions.

ACKNOWLEDGMENT

S.C. is grateful to DAE, Government of India, for support through grants 37/10/95R and DDII/665.

REFERENCES

1. L. J. Gray, J. J, Halliwell, and G. A. Mena Marugan, *Phys. Rev. D* **43**, 2572 (1991).
2. G. A. Mena Marugan, *Class Quant. Grav.* **II**, 589 (1994).
3. A. Ashtkar and R. Tate, *Int. J. Mod. Phys. D* **2**, 15 (1993)
4. A. Ishikawa and H. Veda, *Int. J. Mod. Phys. D* **2**, 249 (1993)
5. S. Chakraborty, *Gen. Rel. Grav.* **29**, 1085 (1997).